Total marks – 120

Attempt All Questions

All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra booklets are available.

Question 1 *(15 Marks)* **Marks**

a) Find $\int \frac{1}{\sqrt{x^2+9}}\,dx$. **1**

b) Use integration by parts to evaluate $\int_1^e \frac{\ln x}{\sqrt{x}}\,dx$ **3**

c) Using the substitution $u = 1 - x$ evaluate $\int_0^{\frac{1}{2}} \frac{x}{(1-x)^2}\,dx$ **3**

d) Find $\int \frac{dx}{x^2+4x+7}$. **2**

e) (i) Show, using a suitable substitution that

$$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx.$$ **2**

(ii) Hence evaluate $\int_0^{\frac{\pi}{4}} \frac{\cos x}{\cos x + \sin x}\,dx$. **4**

Question 2 *(15 Marks)* **Use a SEPARATE writing booklet.** **Marks**

a) Let $z = \dfrac{7-i}{3-4i}$.

(i) Find $|z|$. **2**

(ii) Evaluate $\tan\left\{\tan^{-1}\left(\dfrac{4}{3}\right) - \tan^{-1}\left(\dfrac{1}{7}\right)\right\}$. **2**

(iii) Hence find the principal argument of $\dfrac{7-i}{3-4i}$ in terms of π. **2**

b) The point P represents the complex number z on the Argand diagram. Describe the locus of P when $\arg(z-2) = \arg(z+2) + \dfrac{\pi}{2}$. **2**

c) (i) Assuming the result $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$, and using a suitable substitution, solve the equation $8x^3 - 6x + 1 = 0$. **3**

(ii) Hence find the value of

α) $\cos\dfrac{2\pi}{9} + \cos\dfrac{4\pi}{9} + \cos\dfrac{8\pi}{9}$. **1**

β) $\sec\dfrac{2\pi}{9} + \sec\dfrac{4\pi}{9} + \sec\dfrac{8\pi}{9}$. **3**

Question 3 *(15 Marks)* **Use a SEPARATE writing booklet.** **Marks**

a) Sketch the functions $g(x)=\sqrt{9-x^2}$ and $h(x)=x$ on the same axes. **3**

Use these graphs to sketch $y=f(x)$ where $f(x)=g(x).h(x)$. Hence sketch each of the following on separate number planes.

(i) $y=f(-x)$ **1**

(ii) $y=\dfrac{1}{f(x)}$ **2**

(iii) $y=\left|f(x)\right|$ **1**

(iv) $y^2=f(x)$ **2**

b) (i) Show that $z=i$ is a root of the equation $(2-i)z^2-(1+i)z+1=0$. **1**

(ii) Find the other root of the equation in the form $z=a+ib$, where a and b are real numbers. **2**

c) Let p,q,r be the roots of the equation $x^3-4x+7=0$. Write down the cubic equation in x whose roots are p^2,q^2 and r^2. **3**

Question 4 *(15 Marks)* **Use a SEPARATE writing booklet.** **Marks**

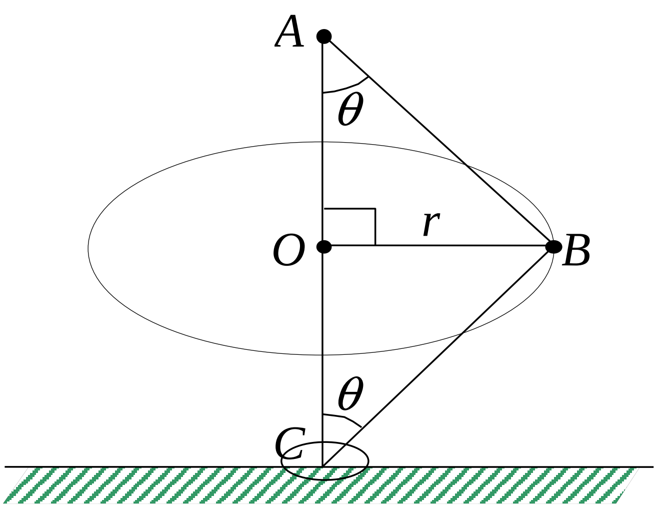
a) A particle of mass 1 kg is projected vertically upwards under gravity with a speed of 2c in a medium which the resistance to motion is $\frac{g}{c^2}$ times the square of the speed, where c is positive constant.

(i) Show that the maximum height (H) reached is **3**

$$H = \frac{c^2}{2g}\ln 5 .$$

(ii) Show that the speed with which the particle returns to its starting point is given by $v = \frac{2c}{\sqrt{5}}$. **4**

b) Two light rigid rods *AB* and *BC*, each of length 0.5 m, are smoothly jointed at *B* and the rod is smoothly jointed at *A* to a fixed smooth vertical rod.



The joint at *B* has a particle of mass 2 kg attached. A small ring of mass 1 kg is smoothly joined to *BC* at *C* and can slide on the vertical rod below *A*. The ring rests on a smooth horizontal ledge at a distance $\frac{\sqrt{3}}{2}$ m below *A*. The system rotates about the vertical rod with constant angular velocity 6 radians per second. Find:

(i) the forces in the rod AB and BC; **5**

(ii) the forces exerted by the ledge on the ring. (let $g = 10m/s^2$) **3**

Question 5 *(15 Marks)* **Use a SEPARATE writing booklet.** **Marks**

a) i) Show that the tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point $P(a\cos\theta, b\sin\theta)$ has the equation $\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$. **3**

ii) This ellipse meets the y-axis at *C* and *D*. Tangents drawn at *C* and *D* on the ellipse meet the tangent in (i) at the points E, F respectively. Prove that $CE.DF=a^2$. **4**

b) i) Show that if $y = mx + k$ is a tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, then $m^2 a^2 - b^2 = k^2$. **3**

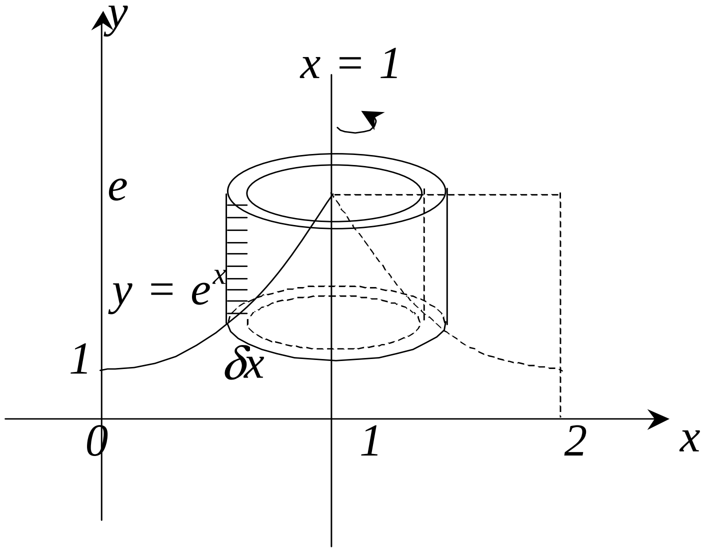
ii) Hence find the equation of the tangents from the point $(1, 3)$ to the hyperbola $\frac{x^2}{4}-\frac{y^2}{15}=1$ and the coordinates of their points of contact. **5**

End of Question 5.

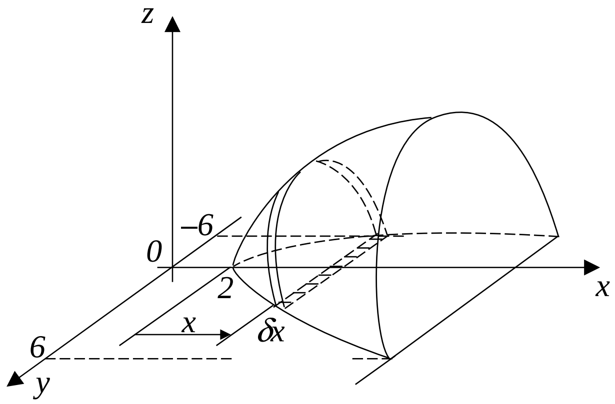
Please Turn Over.

Question 6 *(15 Marks)* **Use a SEPARATE writing booklet.** **Marks**

a) By taking strips parallel to the axis of rotation, use the method of cylindrical shells to find the volume of the solid obtained by rotating the region bounded by $y = e^x$, $y = e$ and the y-axis about the line $x = 1$. **6**



b) The base of a particular solid is the region bounded by the hyperbola $\frac{x^2}{4} - \frac{y^2}{12} = 1$ between its vertex $(2, 0)$ and the corresponding latus rectum. Every cross-section perpendicular to the major axis is a semicircle with diameter in the base of the solid.



i) Find the equation of the latus rectum. **2**

ii) Find the volume of the solid. **4**

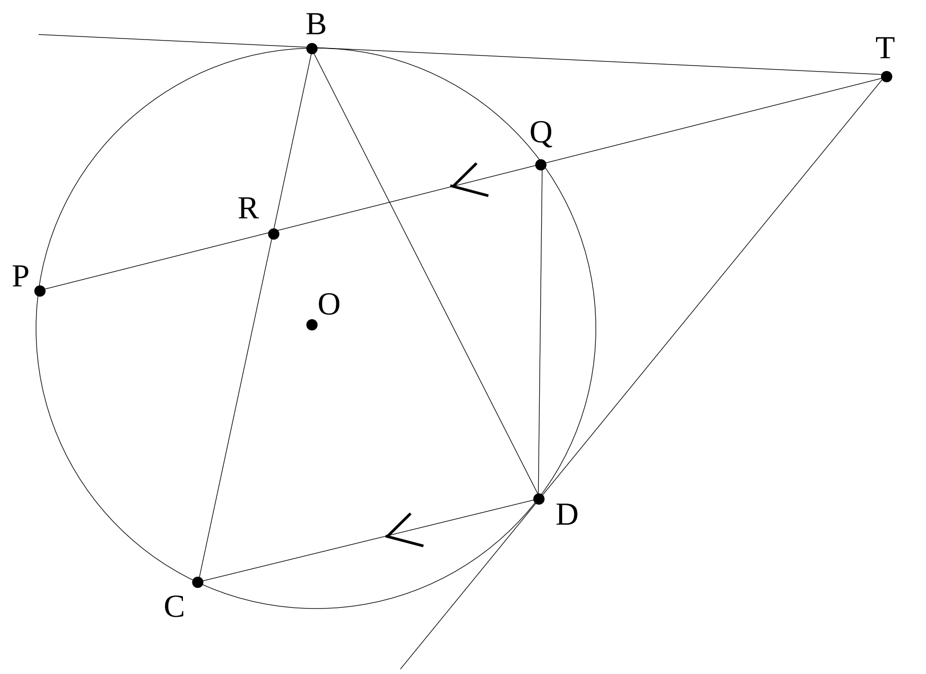
c) The points $P\left(cp, \frac{c}{p}\right)$ and $Q\left(cq, \frac{c}{q}\right)$ lie on the rectangular hyperbola $xy = c^2$. **3**

The chord PQ subtends a right angle at another point $R\left(cr, \frac{c}{r}\right)$ on the hyperbola.

Show that the normal at R is parallel to PQ.

Question 7 *(15 Marks)* **Use a SEPARATE writing booklet.** **Marks**

a)



PQ, CD are parallel chords of a circle, centre O. The tangent at D meets PQ extended at T. B is the point of contact of the other tangent from T. BC meets PQ at R.

(i) Copy the diagram .

(ii) Prove that $\angle$BDT = $\angle$BRT and hence state why B, T, D and R are concyclic points. **3**

(iii) Prove $\angle$BRT = $\angle$DRT. **3**

(iv) Show that ΔRCD is isosceles. **2**

(v) Prove that ΔPRC $\equiv$ ΔQRD. **3**

b) The equation $x^3 + 3px^2 + 3qx + r = 0$, where $p^2 \neq q$, has a double root. Show that **4**

$4\left(p^2 - q\right)\left(q^2 - pr\right) = \left(pq - r\right)^2$.

Question 8 *(15 Marks)* **Use a SEPARATE writing booklet.** **Marks**

a) A coin is tossed six times. What is the probability that there will be more tails on the first three of the six throws than on the last three throws? **3**

b) If *m* points are taken on a straight line and *n* points on a parallel line, how many triangles can be drawn each having its vertices at 3 of the given points? **3**

c) (i) Show that $\left(1-x^2\right)^{\frac{n-3}{2}}-\left(1-x^2\right)^{\frac{n-1}{2}}=x^2\left(1-x^2\right)^{\frac{n-3}{2}}$. **1**

(ii) Let $I_n=\int_0^1\left(1-x^2\right)^{\frac{n-1}{2}}dx$ where $n=0,1,2,....,$ **3**

Show that $nI_n=(n-1)I_{n-2}$ for $n=2,3,4.....$

(iii) Let $J_n=nI_n.I_{n-1}$ for $n=1,2,3,.....$ **3**

By using mathematical induction, prove that

$J_n=\frac{\pi}{2}$ for $n=1,2,3,....$

(iv) Briefly explain why $0<I_n<I_{n-1}$ for $n=1,2,3,....$ **2**

END OF PAPER

Ext 2 Trial 2004.

a) $\int \frac{1}{\sqrt{x^2+9}}\,dx = \ln\left(x + \sqrt{x^2+9}\right) + C$ ✓

b). $\int_1^e \frac{\ln x}{\sqrt{x}}\,dx = \int_1^e \ln x \frac{d}{dx}(2\sqrt{x})\,dx$ ✓

$= \left[2\ln x \times \sqrt{x}\right]_1^e - \int_1^e 2\sqrt{x} \times \frac{1}{x}\,dx$ ✓

$= 2\sqrt{e} - \left[4\sqrt{x}\right]_1^e$ ✓

$= 4 - 2\sqrt{e}.$

c) $\int_0^{\frac{1}{2}} \frac{x}{(1-x)^2}\,dx$ let $u = 1-x \therefore x = 1-u$

$dx = -du$

$x=0 \quad u=1$

$x=\frac{1}{2} \quad u=\frac{1}{2}$ ✓

$= \int_1^{\frac{1}{2}} \frac{(1-u)\times -du}{u^2}$

$= \int_{\frac{1}{2}}^1 \left(\frac{1}{u^2} - \frac{1}{u}\right)du$ ✓

$= \left[-\frac{1}{u} - \ln u\right]_{\frac{1}{2}}^1$

$= -1 - 0 - \left(-2 - \ln\frac{1}{2}\right)$

$= 1 - \ln 2.$ ✓

d) $\int \frac{dx}{x^2+4x+7} = \int \frac{dx}{(x+2)^2+3}$ ✓

$= \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x+2}{\sqrt{3}}\right) + C.$ ✓

e) i). $\int_0^a f(x)\,dx$ let $x = a-u$

$dx = -du.$

$x=0 \Rightarrow u=a$ ✓

$x=a \Rightarrow u=0$

$= \int_a^0 f(a-u)\times -du$ ✓

$= \int_0^a f(a-u)\,du = \int_0^a f(a-x)\,dx$

ii) $\int_0^{\frac{\pi}{4}} \frac{\cos x\,dx}{\cos x + \sin x} = \int_0^{\frac{\pi}{4}} \frac{\cos(\frac{\pi}{4}-x)\,dx}{\cos(\frac{\pi}{4}-x) + \sin(\frac{\pi}{4}-x)}$ ✓

$= \int_0^{\frac{\pi}{4}} \frac{\left(\frac{1}{\sqrt{2}}\cos x + \frac{1}{\sqrt{2}}\sin x\right)dx}{\frac{1}{\sqrt{2}}\cos x + \frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x - \frac{1}{\sqrt{2}}\sin x}$

$= \frac{1}{2}\int_0^{\frac{\pi}{4}} \left(1 + \frac{\sin x}{\cos x}\right)dx$ ✓

$= \frac{1}{2}\left[x - \ln\cos x\right]_0^{\frac{\pi}{4}}$ ✓

$= \frac{1}{2}\left[\frac{\pi}{4} - \ln\frac{1}{\sqrt{2}} - (0-0)\right]$

$= \frac{\pi}{8} + \frac{1}{4}\ln 2.$ ✓

1 a). $z = \frac{7-i}{3-4i} \times \frac{3+4i}{3+4i}$

$= \frac{25+25i}{25} = 1+i$.

(i) $|z| = \sqrt{2}$ or $\frac{|\sqrt{50}|}{|5|} = \sqrt{2}$. ✓✓

ii) let $\alpha = \tan^{-1}\left(\frac{4}{3}\right)$ $\beta = \tan^{-1}\left(\frac{1}{7}\right)$

$\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta}$

$= \frac{\frac{4}{3}-\frac{1}{7}}{1+\frac{4}{3}\times\frac{1}{7}}$ ✓ ✓

$= 1$.

iii) since $z = 1+i$ principal $\arg = \frac{\pi}{4}$ ✓✓

or $\arg\left(\frac{7-i}{3-4i}\right) = \arg(7-i) - \arg(3-4i)$

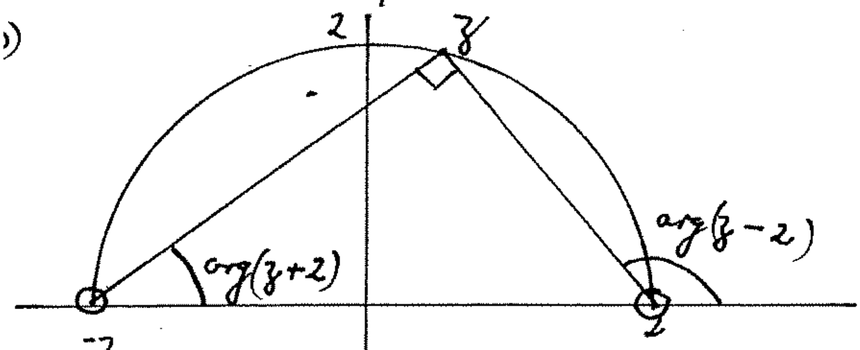
$= \tan^{-1}\left(-\frac{1}{7}\right) - \tan^{-1} -\frac{4}{3}$.

$= \tan^{-1}\left(\frac{4}{3}\right) - \tan^{-1}\left(\frac{1}{7}\right)$

$= \tan^{-1}(1)$ by (ii)

$= \frac{\pi}{4}$.

b)



clearly from diagram $\arg(z-2) = \arg(z+2) + \frac{\pi}{2}$ since exterior angle of △ equal sum of remote interior angles.

∴ Locus of z is the semicircle shown with equation $y = \sqrt{4-x^2}$ for $y > 0$. ✓✓

Note end points are excluded since arg 0 is not defined.

c)(i) let $x = \cos\theta$

$\therefore 8x^3 - 6x + 1 = 0$

$\Rightarrow 2(4\cos^3\theta - 3\cos\theta) = -1$ ✓

$\cos 3\theta = -\frac{1}{2}$ ✓

$3\theta = 2n\pi \pm \frac{2\pi}{3}$. ✓

$\theta = \frac{2\pi}{9}, \frac{4\pi}{9}, \frac{8\pi}{9}$ only 3 soltn since cubic

$\therefore x = \cos\frac{2\pi}{9}, \cos\frac{4\pi}{9}, \cos\frac{8\pi}{9}$ ✓.

ii) α) $\cos\frac{2\pi}{9} + \cos\frac{4\pi}{9} + \cos\frac{8\pi}{9} = \Sigma\alpha = 0$ ✓

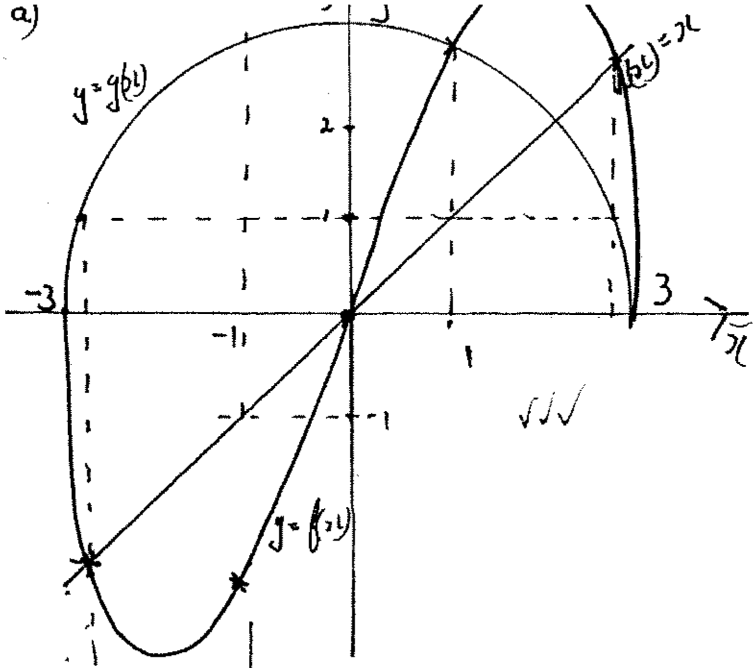
β) $\sec\frac{2\pi}{9} + \sec\frac{4\pi}{9} + \sec\frac{8\pi}{9} = \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$

$= \frac{\Sigma\alpha\beta}{\alpha\beta\gamma}$ ✓

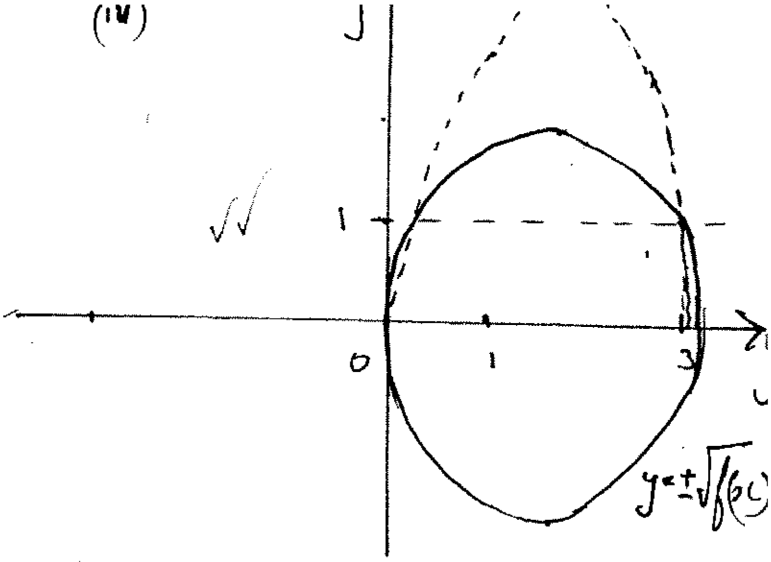
$= \frac{-6/8}{-1/8}$ ✓

$= 6$. ✓

a)



(iv)



b) $P(i) = (2-i)\times -1 - (1+i)i + 1$

$= -2 + i - i + 1 + 1$

$= 0 \therefore z = i$ is a root.

Product of roots: let other root be α.

$\alpha \times i = \frac{1}{2-i}$

$= \frac{2+i}{5}$

× b.s by $-i$

$\alpha = \frac{1}{5} - \frac{2}{5}i$.

c) let $x = \sqrt{y}$.

$y^{3/2} - 4y^{1/2} = -7$.

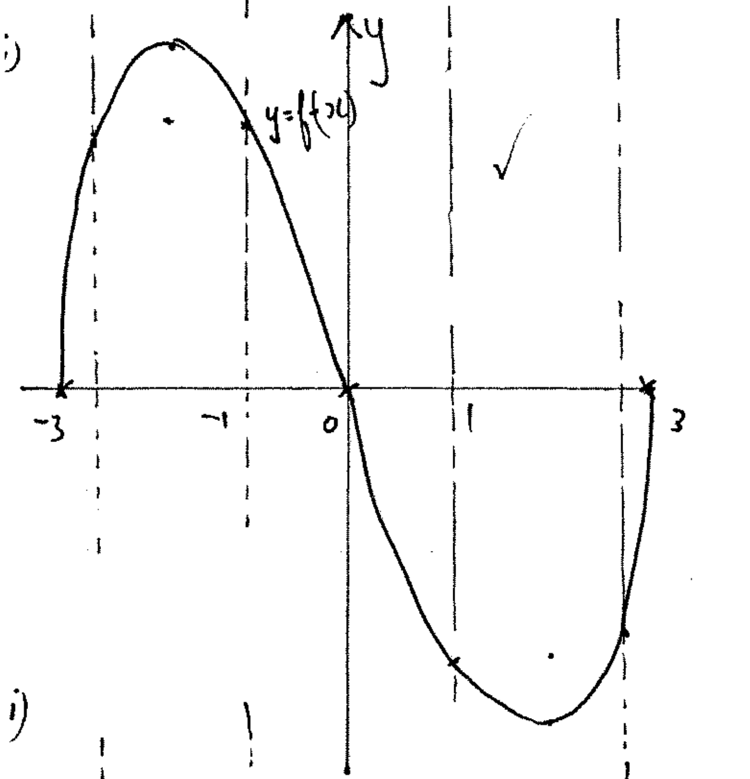
square b.s

$y^3 - 8y^2 + 16y = 49$.

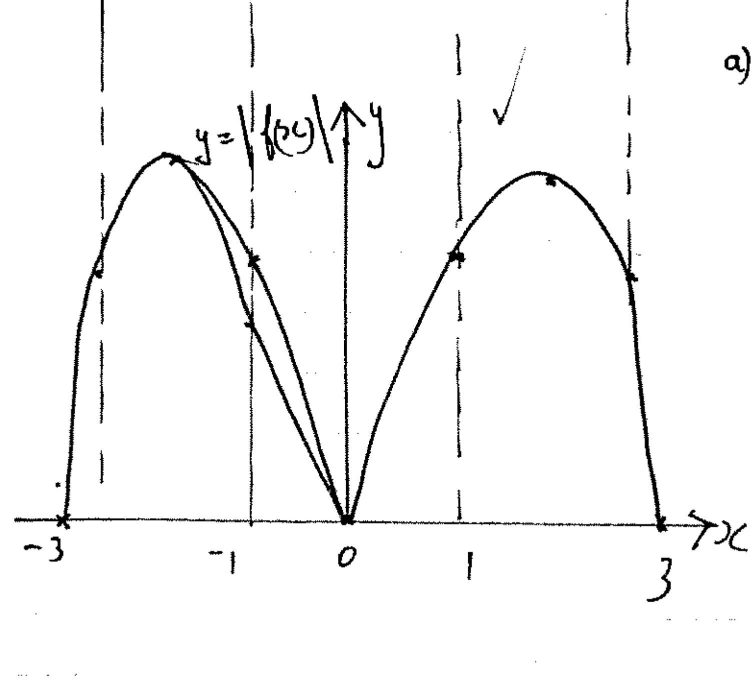
Poly in x

$x^3 - 8x^2 + 16x - 49$.

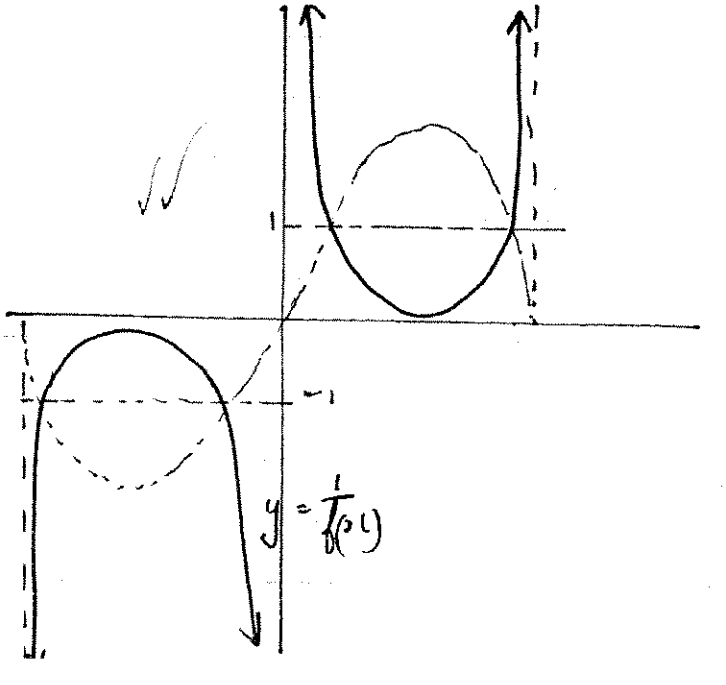
i)



i)



a) ii)



Question 4 *(15 Marks)*

a)

A particle of mass 1 kg is projected vertically upwards under gravity with a speed of 2c in a medium which the resistance to motion is $\frac{g}{c^2}$ times the square of the speed, where c is positive constant.

(i) Show that the maximum height (H) reached is

$$H = \frac{c^2}{2g}\ln 5.$$

SOLUTION:

Upward motion. Choose a point of projection as origin and ↑ as positive.

Initial conditions: $t = 0, x = 0, v = 2c$.

Equation of motion: $\ddot{x} = -g - \frac{g}{c^2}v^2$.

Expression relating x and v:

$$v\frac{dv}{dx} = -g - \frac{g}{c^2}v^2,$$

$$-g\,dx = \frac{v\,dv}{1+\frac{v^2}{c^2}},$$

$$-gx + A = \frac{c^2}{2}\ln\left(1+\frac{v^2}{c^2}\right),\ A \text{ constant;}$$

$$x = 0,\quad v = 2c$$

$$A = \frac{c^2}{2}\ln 5$$

$$x = \frac{c^2}{2g}\ln\frac{5c^2}{c^2+v^2}\ \ldots(1)$$

When the particle reaches its highest point, its velocity is zero.
So $v = 0$

from (2) $t = \frac{c\cdot\tan^{-1}2}{g}$ is the time of ascent.

Let h be the distance between the point of projection and the highest point.
Then $v = 0$ from (1)

$$h = \frac{c^2}{2g}\ln 5.$$

Question 4 a) (ii)

Show that the speed with which the particle returns to its starting point is given by $v = \frac{2c}{\sqrt{5}}$.

SOLUTION:

Downward motion.
Origin at highest point and ↓ as positive direction.
Initial conditions: $t = 0, x = 0, v = 0$.

Equation of motion: $\ddot{x} = g - \frac{g}{c^2}v^2$.

Terminal velocity: as $\ddot{x} \to 0, v \to (c)^- \quad v < c$.
Expression relating x and v:

$$v\frac{dv}{dx} = g - \frac{g}{c^2}v^2$$

$$g\,dx = \frac{v\,dv}{1 - \frac{v^2}{c^2}}$$

$$gx + A = \frac{-c^2}{2}\ln(1 - \frac{v^2}{c^2})$$

, A constant;
$x = 0, v = 0$
$A = 0$

$$x = \frac{c^2}{2g}\ln\frac{c^2}{c^2 - v^2} \quad \dots (2)$$

When the particle returns to its starting point, $x = h$.

Hence from (2) $h = \frac{c^2}{2g}\ln\frac{c^2}{c^2 - v^2}$.

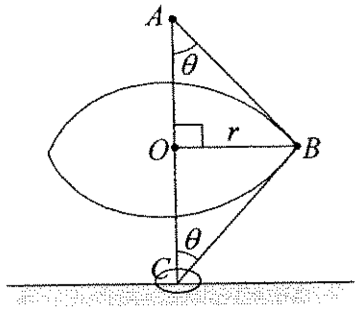
$$h = \frac{c^2}{2g}\ln 5$$

But $5 = \frac{c^2}{c^2 - v^2}$

$$v = \frac{2c}{\sqrt{5}}$$

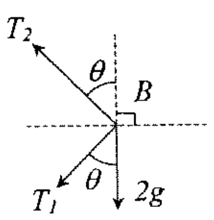
Question 4 b) (i)

SOLUTION:

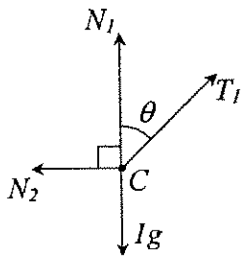


$AB = BC = \frac{1}{2}$, $AC = \frac{\sqrt{3}}{2}$.

Forces on B



Forces on C



N_2 is the force exerted by the rod AC on the ring C, and N_1 is the force exerted by the ledge.

The resultant force on B is $2\omega^2 r$ towards O.

The vertical component is zero

$T_2 \cos\theta - T_1 \cos\theta = 2g$ ...(1)

The horizontal component is $2\omega^2 r$

$T_2 \sin\theta + T_1 \sin\theta = 2\omega^2 r$ ... (2)

But $\omega = 6$, $r = \sqrt{\text{AB}^2 - \text{AO}^2}$

$\cos\theta = \frac{\text{AO}}{\text{AB}}$

$r = \frac{1}{4}$

$\cos\theta = \frac{\sqrt{3}}{2}$, $\sin\theta = \frac{1}{2}$

Hence from (1) and (2) we obtain:

$T_2 - T_1 = \frac{4g}{\sqrt{3}}$ ... (3)

$T_2 + T_1 = 36$ ... (4)

(3) + (4) $T_2 = \frac{2g}{\sqrt{3}} + 18$, $T_2 = \frac{20}{\sqrt{3}} + 18\text{N}$;

(4) − (3)

$T_1 = 18 - \frac{2g}{\sqrt{3}}$

$T_1 = 18 - \frac{20}{\sqrt{3}}$ N

Question 4 b) (ii)

the forces exerted by the ledge on the ring. (let $g = 10m/s^2$)

The resultant force on C is zero.
For its vertical component we have

$N_1 + T_1 \cos\theta = 1g$

$N_1 = g - (18 - \frac{20}{\sqrt{3}})\frac{\sqrt{3}}{2}$

$N_1 = g + 10 - 9\sqrt{3}$

$N_1 = 20 - 9\sqrt{3}$ N

Question 5 *(15 Marks)*

a) i)

Show that the tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point $P(a\cos\theta, b\sin\theta)$ has the equation $\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$.

SOLUTION:

This ellipse meets the y-axis at C and D. Tangents drawn at C and D on the ellipse meet the tangent in (i) at the points E, F respectively. Prove that $CE.DF = a^2$.

SOLUTION:

(i)

Coordinates of C and D are $(0,b)$ and $(0,-b)$ respectively.

$\therefore$ the equations of the tangents through C and D are $y=b$ and $y=-b$, respectively.

Solve each of these equations simultaneously with the equation of the tangent at P,

$\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$ and we have the coordinates of E and F as follows:

$E\left(\frac{a(1-\sin\theta)}{\cos\theta}\right)$ and $F\left(\frac{a(1+\sin\theta)}{\cos\theta}\right)$

$$\begin{aligned}\therefore CE\,.\,DF &= \left(\frac{a(1-\sin\theta)}{\cos\theta}\right).\left(\frac{a(1+\sin\theta)}{\cos\theta}\right)\\ &= \frac{a^2(1-\sin^2\theta)}{\cos^2\theta}\\ &= \frac{a^2\cos^2\theta}{\cos^2\theta}\\ &= a^2\end{aligned}$$

b) i)

Show that if $y = mx + k$ is a tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then $m^2 a^2 - b^2 = k^2$.

SOLUTION: The hyperbola has parametric equations $x = a\sec\theta$ and $y = b\tan\theta$. Hence

$\frac{dy}{dx} = \frac{b\sec\theta}{a\tan\theta}$.

If $y = mx + k$ is a tangent to the hyperbola at $P(a\sec\phi, b\tan\phi)$, then

$m = \frac{dy}{dx}$ at P ... (1)

$ma\tan\phi - b\sec\phi = 0$

P lies on $y = mx + k$

$ma\sec\phi - b\tan\phi = -k$ ... (2)

$(2)^2 - (1)^2$

$m^2a^2(\sec^2\phi - \tan^2\phi) + b^2(\tan^2\phi - \sec^2\phi) = k^2$.

$m^2a^2 - b^2 = k^2$

$(2)\times\sec\phi - (1)\times\tan\phi$

$ma(\sec^2\phi - \tan^2\phi) = -k\sec\phi$

$a\sec\phi = -\frac{ma^2}{k}$,

$(2)\times\tan\phi - (1)\times\sec\phi$

$b(\sec^2\phi - \tan^2\phi) = -k\tan\phi$

$b\tan\phi = -\frac{b^2}{k}$

Therefore the point of contact of the tangent $y = mx + k$ is $P(-\frac{ma^2}{k}, -\frac{b^2}{k})$.

ii)

Hence find the equation of the tangents from the point $(1, 3)$ to the hyperbola $\frac{x^2}{4}-\frac{y^2}{15}=1$ and the coordinates of their points of contact.

SOLUTION:

Now tangents from the point $(1, 3)$ to the hyperbola $\frac{x^2}{4}-\frac{y^2}{15}=1$ have equations of the form $y-3=m(x-1)$, that is, $y=mx+(3-m)$.

Hence $m^2a^2-b^2=k^2$
$4m^2-15=(3-m)^2$
$3m^2+6m-24=0$
$(m-2)(m+4)=0$.

$\therefore m=2$,
$k=3-m=1$,
and $P(-\frac{ma^2}{k},-\frac{b^2}{k})\equiv P(-8,-15)$

or $m=-4$, $k=3-m=7$ and

$P(-\frac{ma^2}{k},-\frac{b^2}{k})\equiv P(\frac{16}{7},-\frac{15}{7})$.

Hence the tangents from the point $(1, 3)$ to the hyperbola $\frac{x^2}{4}-\frac{y^2}{15}=1$ are $y=2x+1$, with point of contact $P(-8,-15)$ and $y=-4x+7$, with point of contact $P(\frac{16}{7},-\frac{15}{7})$.

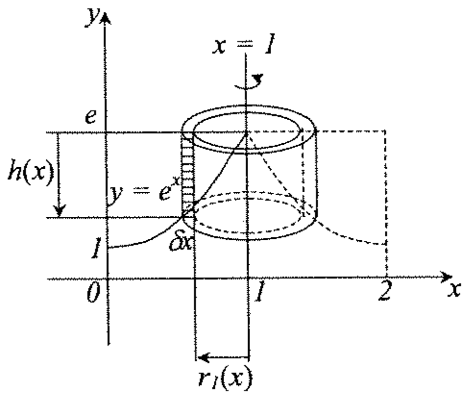
End of Question 5.

Question 6 *(15 Marks)*

a)

By taking strips parallel to the axis of rotation, use the method of cylindrical shells to find the volume of the solid obtained by rotating the region bounded by $y = e^x$, $y = e$ and the y-axis about the line $x = 1$.

SOLUTION:



The typical cylindrical shell has radii $r_1(x) = 1 - x$, $r_2(x) = 1 - x + \delta x$, and height $h(x) = e - e^x$.
This shell has volume

$$\delta V = \pi[(1 - x + \delta x)^2 - (1 - x)^2]h(x)$$
$$= 2\pi(1 - x)(e - e^x)\,\delta x \text{ (ignoring } (\delta x)^2\text{).}$$

$$\therefore\ V = \lim_{\delta x \to 0} \sum_{x=0}^{1} 2\pi(1-x)(e-e^x)\,\delta x$$

$$\therefore V = \int_0^1 2\pi(1-x)(e-e^x)\,dx$$

$$= 2\pi[e\int_0^1 (1-x)\,dx - \int_0^1 (1-x)\,e^x\,dx]$$

$$= 2\pi[e\,(x - \frac{x^2}{2})\Big|_0^1 - \int_0^1 (1-x)\,de^x]$$

$$= 2\pi[\frac{e}{2} - ((1-x)e^x\Big|_0^1 - \int_0^1 (-1)\cdot e^x dx)]$$

$$= 2\pi[\frac{e}{2} + 1 - e^x\Big|_0^1]$$

$$= \pi(4-e)$$

$\therefore$ The volume of the solid is $\pi(4 - e)$ cubic units. »

Question 6

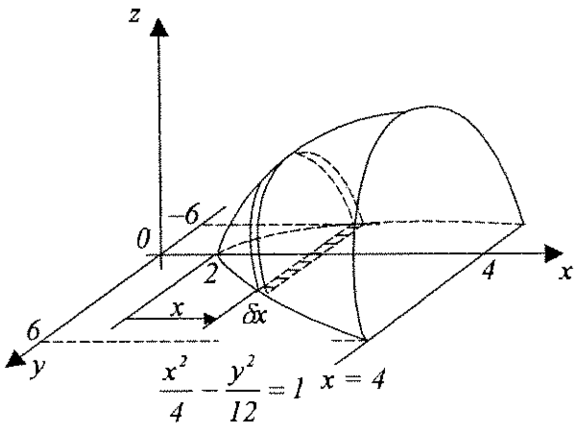
b)

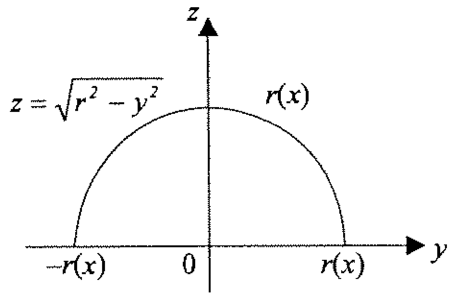
The base of a particular solid is the region bounded by the hyperbola $\frac{x^2}{4}-\frac{y^2}{12}=1$ between its vertex $(2, 0)$ and the corresponding latus rectum. Every cross-section perpendicular to the major axis is a semicircle with diameter in the base of the solid.

i) Find the equation of the latus rectum.

ii) Find the volume of the solid.

SOLUTION:





The latus rectum of the hyperbola $\frac{x^2}{4}-\frac{y^2}{12}=1$ is the line $x = 4$.

The slice is a semicircle with radius r, area of cross-section A and thickness δx.

$$A(x)=\frac{\pi r^2(x)}{2},$$

$$r(x)=\sqrt{12}\cdot\sqrt{\frac{x^2}{4}-1}$$

$$\therefore \quad A(x)=6\pi\left(\frac{x^2}{4}-1\right)$$

The slice has volume $\delta V = A(x)\delta x = 6\pi\left(\frac{x^2}{4}-1\right)\delta x$.

Then the volume of the solid is

$$V=\lim_{\delta x\to 0}\sum_{x=2}^{4}6\pi\left(\frac{x^2}{4}-1\right)\delta x=6\pi\int_2^4\left(\frac{x^2}{4}-1\right)dx$$

$$=6\pi\left(\frac{x^3}{4\cdot 3}-x\right)\Big|_2^4$$

$$=16\pi$$

$\therefore$ The volume of the solid is 16π cubic units. »

Question 6

c)

The points $P\left(cp, \frac{c}{p}\right)$ and $Q\left(cq, \frac{c}{q}\right)$ lie on the rectangular hyperbola $xy = c^2$.

The chord PQ subtends a right angle at another point $R\left(cr, \frac{c}{r}\right)$ on the hyperbola.

Show that the normal at R is parallel to PQ.

SOLUTION:

$xy = c^2$

$x\frac{dy}{dx} + y = 0$

$\frac{dy}{dx} = \frac{-y}{x}$ at $R\left(cr, \frac{c}{r}\right)$

$= \frac{-1}{r^2}$

Hence, gradient of the normal at $R = r^2$

Let gradient of RP = m_{RP}

$$m_{RP} = \frac{\frac{c}{r} - \frac{c}{p}}{cr - cp}$$

$$= \frac{-1}{rp}$$

Similarly $m_{RQ} = \frac{-1}{rq}$ and $m_{PQ} = \frac{-1}{pq}$

Now, $m_{RP} \times m_{RQ} = -1$ $\qquad (\because \angle PRQ = 90^\circ)$

$\therefore \frac{1}{r^2 pq} = -1$

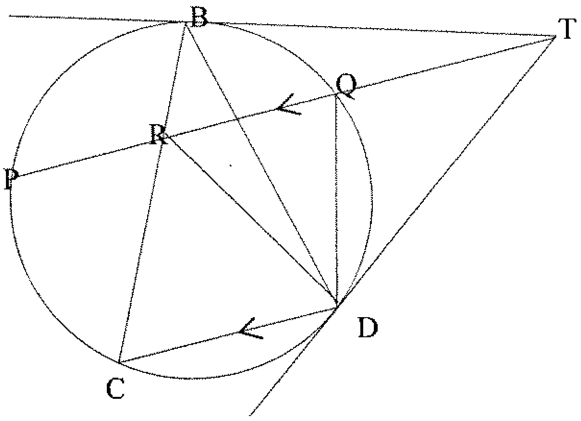
$r^2 = \frac{-1}{pq}$

Hence, gradient of the normal at $R = m_{PQ}$

$\therefore$ Normal at R is parallel to PQ.

Question 7 *(15 Marks)* SOLUTION

a) i)



ii)

Prove that ∠ BDT = ∠ BRT and hence state why B, T, D and R are concyclic points.

SOLUTION:

$\angle BDT = \angle BCD$ (∠ between tangent TD & chord BD = ∠ in Alternate segment)

$\angle BCD = \angle BRT$ (corr. ∠'s, PT || CD)

∴. ∠ BDT = ∠ BRT

Now, as ∠ BDT and ∠ BRT are equal angles subtended by chord BT

∴ BTDR are concyclic points.

iii)

Prove ∠ BRT = ∠ DRT.

SOLUTION I-short version!

In the cyclic quad BTDR

BT = DT (tangents of equal length from external point T)

∴ ∠ BRT = ∠ DRT.(equal chords subtend equal angles to the circumference)

SOLUTION 2

$\angle BTD = 180 - 2 \times \angle BDT$ (∠ sum of triangle BTD)

∴ $\angle BRD = 2 \times \angle BDT$ (opp ∠'s of a cyclic quad are supplementary)

∴ $\angle BRD = 2 \times \angle BRT$ (from (ii) above)

hence ∠ BRT = ∠ DRT.

NOTE: There are many wats of solving this part.

7a)iv) SOLUTION

Show that Δ RCD is isosceles.

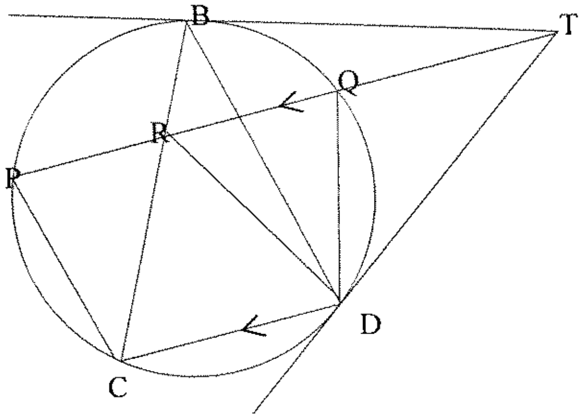
As $\angle BCD = \angle BRT$ (corr. ∠'s, PT||CD)

∠BRT = ∠DRT

∠DRT = ∠RDC (alt. ∠'s, PT||CD)

∴ ∠RDC = ∠BCD

∴ Δ RCD is isosceles (base ∠'s are equal)



v)

Prove that Δ PRC ≡ Δ QRD.

In Δ PRC and Δ QRD

RC=RD (from iv)

∠DRQ=∠RCD (proven above)

∠RCD=∠CRP (alt. ∠'s, PT||CD)

∴ ∠CRP = ∠DRQ

now ∠RQD=180-(∠RCD+∠PCR) (opposite ∠'s of cyclic quad PQDC)

and ∠RPC=180-(∠PRC+∠PCR)

=180-(∠RCD+∠PCR)

∴ ∠RPC = ∠RQD

∴ Δ PRC ≡ Δ QRD.(AAS)

Q7b) SOLUTION

If $ax^3 + bx^2 + d = 0$ has a double root, show that $27a^2d + 4b^3 = 0$.

$P(x) = ax^3 + bx^2 + d,$
$P'(x) = 3ax^2 + 2bx,$
$P''(x) = 6ax + 2b.$

$P'(0) = 0,\ P'(-\frac{2b}{3a}) = 0$. Hence, both 0 and $\frac{-2b}{3a}$ can be a double root of $P(x) = 0$.

Let 0 be a double root.

Hence $P(0) = 0$, $d = 0 \Rightarrow$ if $27a^2d + 4b^3 = 0$, then $b = 0 \Rightarrow P(x) = ax^3$ and 0 is a triple root. Thus if 0 is a double root, then $27a^2d + 4b^3 \neq 0$.

Let $\frac{-2b}{3a}$ be a double root of $P(x) = 0$.

Hence

$$P(\frac{-2b}{3a}) = 0$$

$$a(\frac{-2b}{3a})^3 + b(\frac{-2b}{3a})^2 + d = 0$$

$$27a^2d + 4b^3 = 0$$

Question 8 *(15 Marks)*

a)

A coin is tossed six times. What is the probability that there will be more tails on the first three of the six throws than on the last three throws?

SOLUTION

3 outcomes: Equal tails, more tails or less tails.

P(equal tails)=P(1H) + P(2H) + P(3H) + P(0H)

$$=9\left(\frac{1}{2}\right)^6+9\left(\frac{1}{2}\right)^6+\left(\frac{1}{2}\right)^6+\left(\frac{1}{2}\right)^6$$

$$=\frac{20}{64}$$

P(More tails in 1st 3 throws)

$$=\frac{1}{2}\left(1-\frac{20}{64}\right)$$

$$=\frac{11}{32}$$

b)

If m points are taken on a straight line and n points on a parallel line, how many triangles can be drawn each having its vertices at 3 of the given points?

SOLUTION

Number of triangles

$$=m\,{}^nC_2+n\,{}^mC_2$$

$$=\frac{1}{2}mn(m+n-2)$$

Question 8c)

(i)

Show that $\left(1-x^2\right)^{\frac{n-3}{2}}-\left(1-x^2\right)^{\frac{n-1}{2}}=x^2\left(1-x^2\right)^{\frac{n-3}{2}}$.

SOLUTION

$$\left(1-x^2\right)^{\frac{n-3}{2}}-\left(1-x^2\right)^{\frac{n-1}{2}}=x^2\left(1-x^2\right)^{\frac{n-3}{2}}$$

$$=\left(1-x^2\right)^{\frac{n-3}{2}}\left[1-\left(1-x^2\right)^{\frac{2}{2}}\right]$$

$$=x^2\left(1-x^2\right)^{\frac{n-3}{2}}$$

(ii)

SOLUTION

Using Integration by parts;

$$I_n=\int_0^1\left(1-x^2\right)^{\frac{n-1}{2}}\frac{d(x)}{dx}dx$$

$$=\left[x\left(1-x^2\right)^{\frac{n-1}{2}}\right]_0^1-\frac{n-1}{2}\int_0^1-2x^2\left(1-x^2\right)^{\frac{n-3}{2}}dx$$

$$\therefore I_n=(n-1)\int_0^1 x^2\left(1-x^2\right)^{\frac{n-3}{2}}dx$$

now from $(c)i$

$$I_n=(n-1)\int_0^1 x^2\left(1-x^2\right)^{\frac{n-3}{2}}dx$$

$$I_n=(n-1)\int_0^1\left(1-x^2\right)^{\frac{n-3}{2}}dx-(n-1)\int_0^1\left(1-x^2\right)^{\frac{n-1}{2}}dx$$

$$I_n=(n-1)I_{n-2}-(n-1)I_n$$

$$\therefore (n-1)I_n+I_n=(n-1)I_{n-2}$$

~~Let~~ $\therefore nI_n=(n-1)I_{n-2}$

Question 8 (iii)

Let $J_n = nI_n.I_{n-1}$ for $n = 1, 2, 3, \ldots.$

By using mathematical induction, prove that

$J_n = \frac{\pi}{2}$ for $n = 2, 3, \ldots$

SOLUTION

Test for $n=2$

$$J_2 = 2I_2.I_{2-1}$$
$$= I_0.I_1$$

$$I_2 = \int_0^1 \left(1-x^2\right)^{\frac{-1}{2}} dx. \int_0^1 \left(1-x^2\right)^0 dx$$

$$= \int_0^1 \left(1-x^2\right)^{\frac{-1}{2}} dx.$$

$$= \left[\sin^{-1} x\right]_0^1$$

$$= \frac{\pi}{2}$$

Assume true for $n=k$ *ie* $J_k = kI_k.I_{k-1} = \frac{\pi}{2}$

Test for $n=k+1$

Now from $J_n = nI_n.I_{n-1}$

$$J_{k+1} = (k+1)I_{k+1}.I_k$$

And as $nI_n = (n-1)I_{n-2}$

therefore $(k+1)I_{k+1} = kI_{k-1}$

$$J_{k+1} = kI_{k-1}.I_k$$
$$= \frac{\pi}{2}$$

Hence by Mathematical Induction

$J_n = \frac{\pi}{2}$ for $n = 1, 2, 3, \ldots$

(iv)

Briefly explain why $0 < I_n < I_{n-1}$ for $n = 1, 2, 3, \ldots$

SOLUTION

$$I_n = \int_0^1 (1-x^2)^{\frac{n-1}{2}}\,dt > 0 \quad \text{clearly!}$$

$$I_{n-1} = \int_0^1 (1-x^2)^{\frac{n-3}{2}}\,dx$$

$$\therefore I_n - I_{n-1} = \int_0^1 \left[(1-x^2)^{\frac{n-1}{2}} - (1-x^2)^{n-3}\right] dx$$

$$= \int_0^1 (1-x^2)^{\frac{n-2}{2}}\left[\sqrt{1-x^2} - 1\right] dx$$

for $0 \le x \le 1$

$0 \le \sqrt{1-x^2} \le 1$

$\therefore -1 \le \sqrt{1-x^2} - 1 \le 0$

Hence $I_n - I_{n-1} < 0$

$\therefore 0 < I_n < I_{n-1}$

END OF PAPER